ZNOTES.ORG

UPDATED TO 2020-21 SYLLABUS

CAIE AS LEVEL MATHS (9709)

SUMMARIZED NOTES ON THE PURE 1 SYLLABUS

1. Quadratics

1.1. Completing the square

$$x^2 + nx$$

$$x^2 + nx \Longleftrightarrow$$

$$(x + \frac{n}{2})^2 - (\frac{n}{2}^2)$$

$$\frac{n}{2}$$

$$a\left(x+n\right)^2 + k$$

Where the vertex is $(-n, k)$

1.2. Sketching the Graph

- y-intercept
- x-intercept
- Vertex (turning point)

1.3. Discriminant

$$b^2 - 4\text{ac}$$

If $b^2 - 4\text{ac} = 0$, real and equal (repeated) roots
If $b^2 - 4\text{ac} < 0$, no real roots
If $b^2 - 4\text{ac} > 0$, real and distinct roots

1.4. Quadratic Inequalities

Case 1: Assuming $d < \beta$,

$$(x-d)(x-\beta) < 0 \Longrightarrow d < x < \beta$$

$$(x-d)(x-\beta) > 0 \Longrightarrow x < d \; or \; x > \beta$$

Case 2: When no x coefficient,

$$x^2 - c > 0$$

$$\Longrightarrow x < -\sqrt{c} \;\; or \;\; x > \sqrt{c}$$

$$x^2 - c \leq 0$$

$$\Longrightarrow -\sqrt{c} \leq x \leq \sqrt{c}$$

1.5. Solving Equations in Quadratic Form

- To solve an equation in some form of quadratic.
- Substitute by another variable.
- E.g. $2x^4 + 3x^2 + 7$, use $u = x^2, \therefore 2u^2 + 3u + 7$

2. Functions

Domain = x values & Range = y values

- Function: mapping of an x-value to a y-value

2.2. Find Range

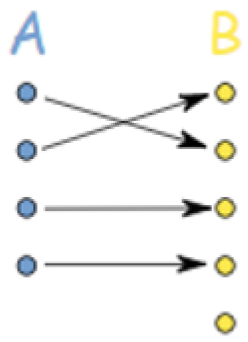
- Find the highest possible y-value and lowest possible y-value based on the domain
- For Quadratic functions, such as $f\left(x\right) = 3x^2 + 5x - 6$, complete square first to find vertex and use it to find its range.
  - If coefficient of x^2 is positive, vertex is minimum
  - If coefficient of x^2 is negative, vertex is maximum

2.3. Composition of 2 Functions

- Definition: a function with another function as an input $\text{fg}\left(x\right) \Rightarrow f\left(g\left(x\right)\right)$
- E.g. $f\left(x\right) = 4x + 5 \quad g\left(x\right) = x^2 - 5$
  - Then $\text{fg}\left(x\right) = 4\left(x^2 - 5\right) + 5$
- A composite function like $\text{fg}\left(x\right)$ can only be formed when the range of $g\left(x\right)$ is within the domain of $f\left(x\right)$

2.4. One-One Functions

- Definition: One x value substitutes to give one y value



- No indices
- If function is not one-to-one, restrict the function in a domain such that the function is one-to-one under that domain.
- Only one-to-one functions are invertible

2.5. Finding Inverse

- Definition: An inverse function shows what the input is based from the output e.g. if $f\left(3\right) = 5$ then $f^{-1}\left(5\right) = 3$. In other words, it reverses the process. The graph of $y = f\left(x\right)$ and $y = f^{-1}\left(x\right)$ is symmetrical by the line $y = x$.
- An inverse function has a property such that:

$$ff^{-1}\left(x\right) = f^{-1}f\left(x\right) = x$$

Make sure that it is a one-to-one function if it is then,

- Write $f\left(x\right)$ as y
- Make x the subject
- Swap every single x with y. By now you should have y as the subject
- Replace y with $f^{-1}\left(x\right)$. Read as "The f inverse of x"

Example:

$$f\left(x\right) = 3x + 4$$

$$y = 3x + 4$$

$$y - 4 = 3x$$

$$x = \frac{y-4}{3}$$

Swap all the x with y,

$$y = \frac{x-4}{3}$$

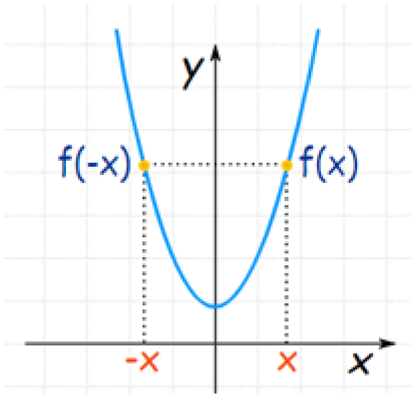
Replace y with $f^{-1}\left(x\right)$,

$$f^{-1}\left(x\right) = \frac{x-4}{3}$$

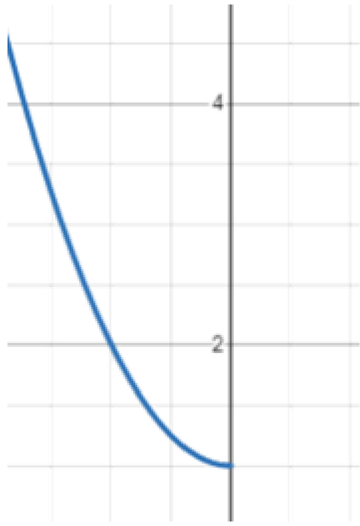
Example:
Make $f\left(x\right) = x^2 + 1$ a one-to-one function.
Solution:

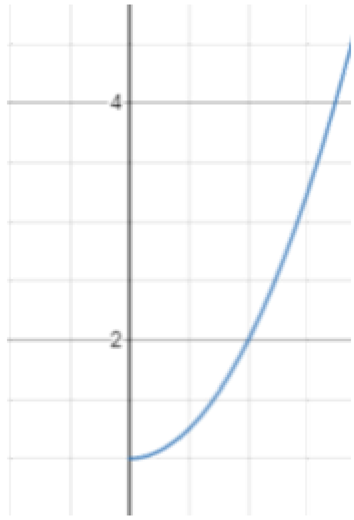
$$\mathbf{x^2+1}, \quad -\infty < x < \infty$$

One value of $\mathbf{x}$ that doesn't have alternate value of $\mathbf{x}$ which maps same value of $\mathbf{y}$ is $\mathbf{0}$
∴ We separate the function into two functions



$$\mathbf{x^2+1,\ x \le 0 \text{ and } x^2+1,\ 0 \le x}$$





2.6. Relationship of Function & its Inverse

- The graph of the inverse of a function is the reflection of a graph of the function in $y = x$

{W12-P11} Question 10:
$f\left(x\right) = 4x^2 - 24x + 11$, for $x \in \mathbb{R}$
$g\left(x\right) = 4x^2 - 24x + 11$, for $x \le 1$

1. Express $f(x)$ in the form $a\left(x-b\right)^2 + c$, hence state coordinates of the vertex of the graph $y = f(x)$
2. State the range of g
3. Find an expression for $g^{-1}(x)$ and state its domain

Solution:

Part (i)
First pull out constant, 4, from x related terms:

$$4\left(x^2 - 6x\right) + 11$$

Use following formula to simplify the bracket only:

$$\left(x - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

$$4\left[(x-3)^2-3^2\right]+11$$

$$4(x-3)^2-25$$

Part (ii)
Observe given domain, $x \leq 1$.
Substitute highest value of x

$$g(x)=4(1-3)^2-25=-9$$

Substitute next 3 whole numbers in domain:

$$x=0,\ -1,\ -2 \qquad g(x)=11,\ 23,\ 75$$

Thus, they are increasing

$$\therefore g(x) \geq -9$$

Part (iii)
Let $y=g(x)$, make x the subject

$$y=4(x-3)^2-25$$

$$\frac{y+25}{4}=(x-3)^2$$

$$x=3+\sqrt{\frac{y+25}{4}}$$

Can be simplified more

$$x=3\pm\frac{1}{2}\sqrt{y+25}$$

Positive variant is not possible because $x \leq 1$ and using positive variant would give values above 3

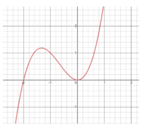
$$\therefore x=3-\frac{1}{2}\sqrt{y+25}$$

$$\therefore g^{-1}(x)=3-\frac{1}{2}\sqrt{x+25}$$

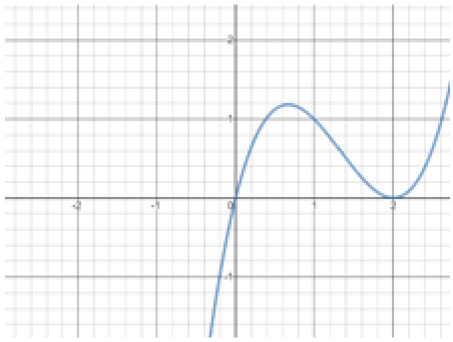
Domain of $g^{-1}(x)=$ Range of g $(x) \therefore x \geq -9$

2.7. Translation

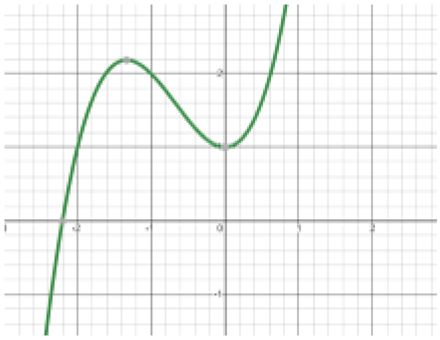
- Let $y=f(x)$



- Shift along x-axis by a units to the right: $f(x-a)$

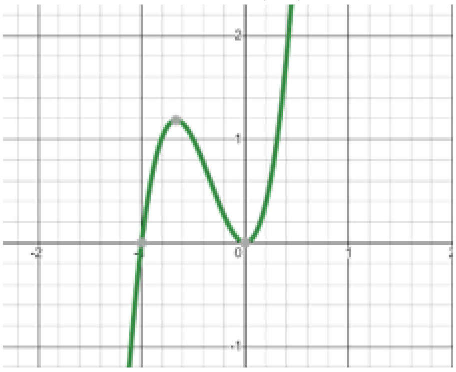


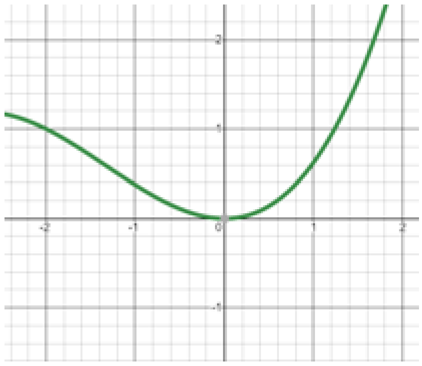
- Shift along y-axis by b units upwards: $f(x)+b$



2.8. Stretch

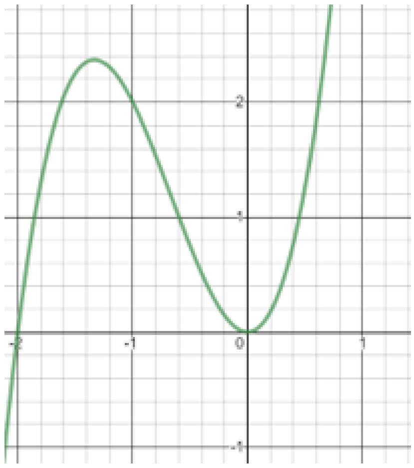
- Stretches the graph sideways: $f(\mathrm{ax})$

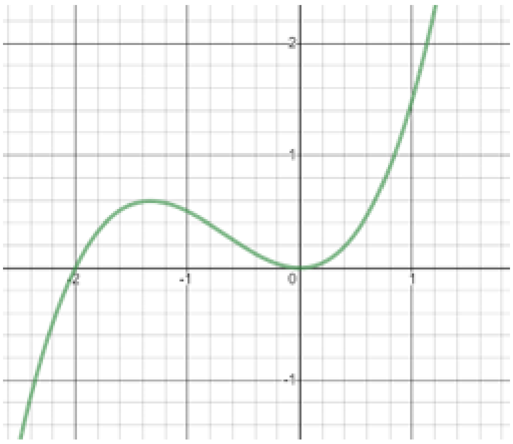




  - If $a>1$ it will **shrink** the graph sideways
  - If $0<a<1$ it will **expand** the graph sideways

- Stretches upwards and downwards: $\mathrm{af}(x)$





- If $a > 1$ it will **expand** the graph up & downwards
- If $0 < a < 1$ it will **shrink** the graph up & downwards

3. Coordinate Geometry

3.1. Length of a Line Segment

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3.2. Midpoint of a Line Segment

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

3.3. Equation of a Straight Line

- $y = \text{mx} + c$
- $y - y_1 = m(x - x_1)$

3.4. Special Gradients

- Parallel lines: $m_1 = m_2$
- Perpendicular lines: $m_1 \times m_2 = -1$
- The gradient at any point on a curve is the gradient of the tangent to the curve at that point
- The gradient of a tangent at the vertex of a curve is equal to zero – stationary point

{S13-P12} Question 7:

Point R is a reflection of the point $(-1, 3)$ in the line $3y + 2x = 33$.

Find by calculation the coordinates of R

Solution:

Find the equation of line perpendicular to $3y + 2x = 33$ intersecting point $(-1, 3)$

$$3y + 2x = 33 \Leftrightarrow y = 11 - \frac{2}{3}x$$

$$m = -\frac{2}{3}$$

$m \times m_1 = -1$ and so $m_1 = \frac{3}{2}$

Perpendicular general equation:

$$y = \frac{3}{2}x + c$$

Substitute known values

$3 = \frac{3}{2}(-1) + c$ and so $c = \frac{9}{2}$

Final perpendicular equation:

$$2y = 3x + 9$$

Find the point of intersection by equating two equations

$$11 - \frac{2}{3}x = \frac{3x + 9}{2}$$

$$13 = \frac{13}{3}x$$

$$x = 3, \quad y = 9$$

Vector change from $(-1, 3)$ to $(3, 9)$ is the vector change from $(3, 9)$ to R

Finding the vector change:

$$Change\ in\ x = 3 - -1 = 4$$

$$Change\ in\ y = 9 - 3 = 6$$

Thus R

$x = 3 + 4 = 7$ and $y = 9 + 6 = 15$

$$R = (7, 15)$$

3.5. Equation of a circle

- Standard Form: $(x - a)^2 + (y - b)^2 = r^2$
  - Centre = (a, b)
  - Radius = r
- General form: $x^2 + y^2 + \text{ax} + \text{by} + c = 0$
  - Centre = $\left(\frac{-a}{2}, \frac{-b}{2}\right)$
  - Radius = $\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c^2$
  - Note: if eqn. of circle is in general form, it's highly recommended to convert it into its standard form by completing square to easily find center and radius

- Tangents on a circle are **always** perpendicular to its radius
- If a **right-angled triangle** is inscribed in a circle, its hypotenuse is the diameter of the circle

Example
The equation of a circle: $x^2+y^2+4x+2y-20=0$ The line L has the equation $7x+y=10$ intersects the circle at point A and B. The x-coordinate of A is less than the x-coordinate of B.

1. Find the center and the length of diameter of the circle
2. Find the coordinates of A and B

Solution:
i. Rearrange the equation to standard form by using completing square:

$$x^2+4x+y^2+2y=20$$

$$(x+2)^2-4+(y+1)^2-1=20$$

$$\Rightarrow (x+2)^2+(y+1)^2=25$$

$\therefore$ its center: $(-2,-1)$. Its diameter: $2\times 5 \ = \ 10$
ii.Do simultaneous equation

$$(x+2)^2+(y+1)^2=25 \ \ \& \ \ y=-7x+10$$

Use substitution $y=-7x+10$ onto $(x+2)^2+(y+1)^2=25$.

$$(x+2)^2+(-7x+11)^2=25$$

Find x

$$x^2+4x+4+49x^2-154x+121=25$$

$$50x^2-150x+100=0$$

$$x^2-3x+2=0$$

$$\therefore x=1 \ \ x=2$$

Put x values back into $y=-7x+10$ to find y value:

$$\therefore A\,(1,3) \ \ \mathrm{B}\,(2,-4)$$

4. Circular Measure

4.1. Radians

$\pi=180°$ and $2\pi=360°$
Degrees to radians: $\theta\times\frac{\pi}{180}$
Radians to degrees: $\theta\times\frac{180}{\pi}$

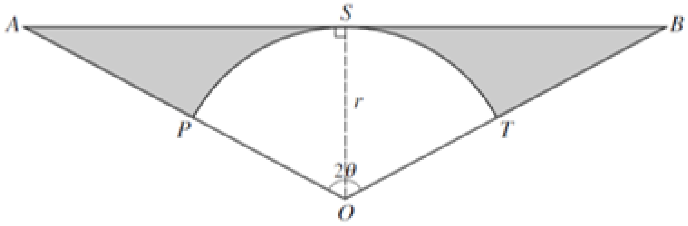
4.2. Arc length

$s=r\theta$ In Radians

4.3. Area of a Sector

$$A=\frac{1}{2}r^2\theta \qquad\qquad \text{In Radians}$$

{S11-P11} Question 9:



Triangle OAB is isosceles, $OA=OB$ and ASB is a tangent to PST

1. Find the total area of the shaded region in terms of r and π
2. When $\theta=\frac{\pi}{3}$and $r=6$, find the total perimeter of the shaded region in terms of $\sqrt{3}$ and π

Solution:
Part (i)
Use trigonometric ratios to form the following:

$$\mathrm{AS}=r\tan\theta$$

Find the area of triangle OAS:

$$\mathrm{OAS}=\frac{r\tan\theta\times r}{2}=\frac{1}{2}r^2\tan\theta$$

Use the formula of the sector to find the area of OPS:

$$OPS=\frac{1}{2}r^2\theta$$

Area of ASP is $OAS-OPS$:

$$\therefore ASP=\frac{1}{2}r^2\tan\theta-\frac{1}{2}r^2\theta=\frac{1}{2}r^2\,(\tan\theta-\theta)$$

Multiply final by 2 because BST is the same and shaded is ASP and BST

$$Area=2\times\frac{1}{2}r^2\,(\tan\theta-\theta)= \ r^2\,(\tan\theta-\theta)$$

Part (ii)
Use trigonometric ratios to get the following:

$$\cos\left(\frac{\pi}{3}\right)=\frac{6}{\mathrm{AO}}$$

$$\therefore AO=12$$

Finding AP:

$$AP=AO-r=12-6=6$$

Finding AS:

$$AS = 6\tan\left(\frac{\pi}{3}\right) = 6\sqrt{3}$$

Finding arc PS:

$$Arc\ PS = r\theta$$

$$PS = 6 \times \frac{\pi}{3} = 2\pi$$

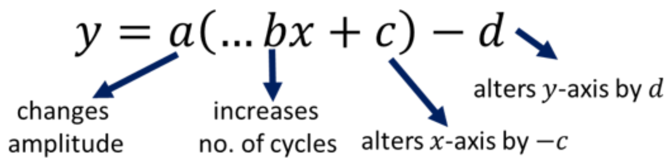
The perimeter of 1 side of the shaded region:

$$\text{Pe}_1 = 6 + 6\sqrt{3} + 2\pi$$

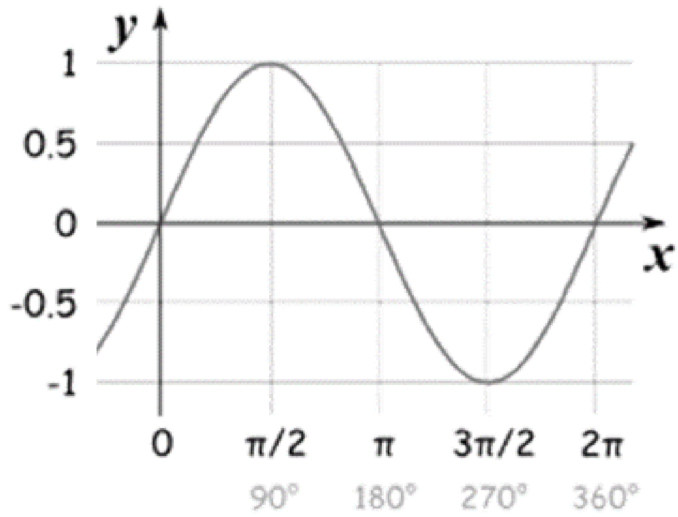
Perimeter of the entire shaded region is double:

$$2 \times \text{Pe}_1 = 12 + 12\sqrt{3} + 4\pi$$

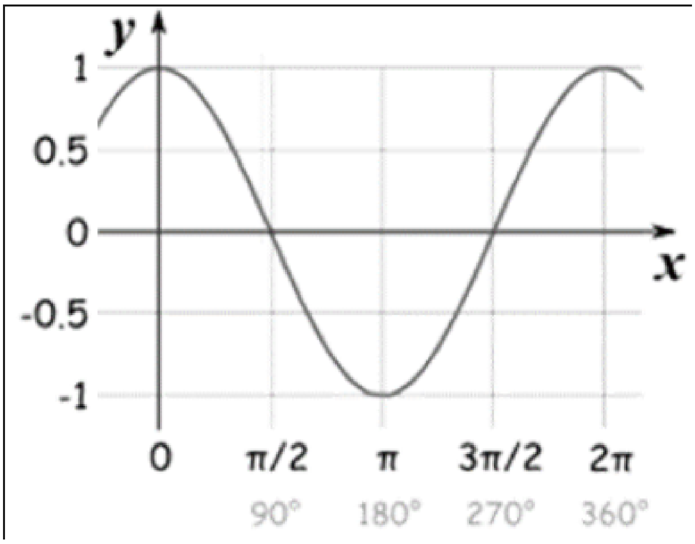
5. Trigonometry



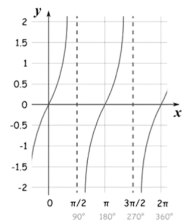
5.2. Sine Curve



5.3. Cosine Curve



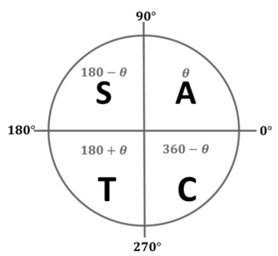
5.4. Tangent Curve



5.5. Exact values of Trigonometric Functions

Angle (θ)		$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
Degrees	Radians			
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Not Defined

5.6. When sin, cos and tan are positive



5.7. Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

5.8. Inverse Functions

- If $\text{trig}(\theta) = a$, then $\theta = \text{trig}^{-1}(a)$
  - Where "trig" represents any Trigonometric Function
  - Inverse trigonometric functions are used to find angle

6. Series

6.1. Binomial Expansion

- A neat way of expanding terms with high powers.

$$(x+y)^n = nC_0x^n + nC_1x^{n-1}y + nC_2x^{n-2}y^2 + \ldots + nC_ny^n$$

$$nC_r = \frac{n!}{r!(n-r)!}$$

$$\text{In summation}: (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$$

(The summation form is just another way to express $(a+b)^n$, it's not important but some students may like to see it that way)

6.2. Arithmetic Progression

- Definition: Sequence where successive terms are gained from adding same value E.g. 1,3,5,7,9,11...
- u_n = a + (n − 1)d
- $s_n = \frac{1}{2}n[2a + (n-1)d]$
  - u_n = the n-th term of the sequence
  - a = First term of the sequence
  - n = The n-th term o d = Main difference
  - s_n = Sum from 1st term to n-th term

6.3. Geometric Progression

- Definition: Sequence where successive terms are gained from multiplying the same value E.g. 2,4,8,16,32...

$$u_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

- u_n = the n-th term of the sequence
- a = First term of the sequence
- n = The n-th term
- r = Common Ratio
- S_n = Sum from 1st term to n-th term

When $|r| < 1$, Sum to infinity:

$$S_\infty = \frac{a}{1-r}$$

{W05-P01} Question 6:
A small trading company made a profit of 250000 dollars in the year 2000. The company considered two different plans, plan A and plan B, for increasing its profits. Under plan A, the annual profit would increase each year by 5% of its value in the preceding year. Under plan B, the annual profit would increase each year by a constant amount of D

1. Find for plan A, the profit for the year 2008
2. Find for plan A, the total profit for the 10 years 2000 to 2009 inclusive
3. Find for plan B the value of D for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for plan A

Solution:
Part (i)
Increases are exponential ∴ it is a geometric sequence:
2008 is the 9th term:
$\therefore u_9 = 250000 \times 1.05^{9-1} = 369000$ (3s.f.)
Part (ii)
Use sum of geometric sequence formula:

$$S_{10} = \frac{250000(1-1.05^{10})}{1-1.05} = 3140000$$

Part (iii)
Plan B arithmetic; equate 3140000 with sum formula

$$3140000 = \frac{1}{2}(10)(2(250000) + (10-1)D)$$

$$D = 14300$$

7. Differentiation

When $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

- 1st Derivative = $\frac{dy}{dx} = f'(x)$
- 2nd Derivative = $\frac{d^2y}{dx^2} = f''(x)$
- Increasing function: $\frac{dy}{dx} > 0$
- Decreasing function: $\frac{dy}{dx} < 0$
- Stationary point: $\frac{dy}{dx} = 0$

7.2. Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$(f(g(x)))' = f'(g(x)) \times g'(x)$$

Example
Differentiate $y = (x + x^3)^5$
Solution:
Let $u = x + x^3$, then find $\frac{du}{dx}$

$$u = x + x^3$$

$$\frac{du}{dx} = 1 + 3x^2$$

Now $y = u^5$

$$\frac{\mathrm{dy}}{\mathrm{du}} = 5u^4$$

Multiply them together

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{du}} \times \frac{\mathrm{du}}{\mathrm{dx}} = \left(1+3x^2\right) \times 5\left(x+x^3\right)^4$$

Another quick way is to:
1. Take the derivative of the "inside"
2. Then take the derivative of the "outside"
3. Multiply them together

In our case:
The inside: $x + x^3$
The outside: u^5
So, differentiating will give us $\left(1+3x^2\right) \times 5\left(x+x^3\right)^4$

7.3. Connected Rates of Change

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}} \Big/ \frac{\mathrm{dx}}{\mathrm{dt}} \text{ or } \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}} \times \frac{\mathrm{dt}}{\mathrm{dx}}$$

{W05-P01} Question 6:
The equation of a curve is given by the formula:

$$y = \frac{6}{5-2x}$$

1. Calculate the gradient of the curve at the point where $x = 1$
2. A point with coordinates (x, y) moves along a curve in such a way that the rate of increase of y has a constant value of 0.02 units per second. Find the rate of increase of x when $x = 1$

Solution:
Part (i)
Differentiate given equation

$$6\left(5-2x\right)^{-1}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = 6\left(5-2x\right)^{-2} \times -2 \times -1$$

$$= 12\left(5-2x\right)^{-2}$$

Now we substitute the given x value:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = 12\left(5-2\left(1\right)\right)^{-2}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{4}{3}$$

Thus, the gradient is equal to $\frac{4}{3}$ at this point
Part (ii)
Rate of increase in time can be written as:

$$\frac{\mathrm{dx}}{\mathrm{dt}}$$

We know the following:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{4}{3} \qquad and \qquad \frac{\mathrm{dy}}{\mathrm{dt}} = 0.02$$

Thus, we can formulate an equation:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{dy}}{\mathrm{dt}} \div \frac{\mathrm{dx}}{\mathrm{dt}}$$

Rearranging the formula, we get:

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \frac{\mathrm{dy}}{\mathrm{dt}} \div \frac{\mathrm{dy}}{\mathrm{dx}}$$

Substitute values into the formula

$$\frac{\mathrm{dx}}{\mathrm{dt}} = 0.02 \div \frac{4}{3}$$

$$\frac{\mathrm{dx}}{\mathrm{dt}} = 0.02 \times \frac{3}{4} = 0.015$$

7.4. Nature of Stationary Point

- Find second derivative $\frac{d^2y}{dx^2}$
  - Substitute x-value of stationary point
  - If value +ve → min. point, $\frac{d^2y}{dx^2} > 0$
  - If value –ve → max. point $\frac{d^2y}{dx^2} < 0$

8. Integration

$$\int ax^n \,\mathrm{dx} = \frac{ax^{n+1}}{n+1} + c$$

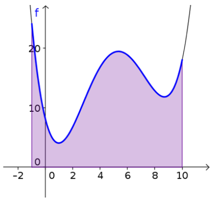
$$\int \left(ax+b\right)^n \,\mathrm{dx} = \frac{\left(ax+b\right)^{n+1}}{a\left(n+1\right)} + c$$

- Integration is the reverse process of differentiation
- The "S" shaped symbol is used to mean the integral of, and dx is written at the end of the terms to be integrated, meaning "with respect to x". This is the same "dx" that appears in $\frac{\mathrm{dy}}{\mathrm{dx}}$.
- Indefinite Integrals: Integrals without limits of integration (the numbers by the integral sign), **don't forget to include +c**
- Definite Integrals: Integrals with limits of integration, no need of putting +c
- Use coordinates of a point on the curve to find **c** when integrating a derivative to find equation of the curve.

8.2. Area Under a Curve

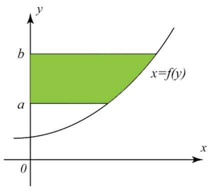
- Area bounded by the curve to the x-axis
  - This is the most common integrals being used
  - Use dx
  - Make y the subject in the equation then input it into your integral

$$\int_a^b \mathrm{y\ dx}$$



- Area bounded by the curve to the y-axis
  - Use dy
  - Make x the subject of the equation and then input it into the integral

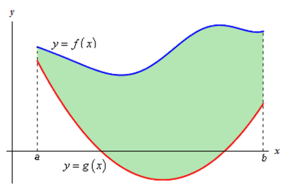
$$\int_a^b x\ \mathrm{dy}$$



8.3. Area Between Two Curves

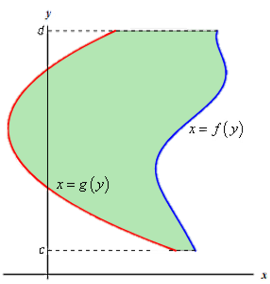
- Area between two curves with respect to x
  - Just like finding the area under a curve, this time you subtract the first curve by the second curve
  - Use dx
  - Make sure both equations have y as the subject

$$\int_a^b y_1 - y_2\ \mathrm{dx} \quad \text{or} \quad \int_a^b y_1\ dx - \int_a^b y_2\ \mathrm{dx}$$

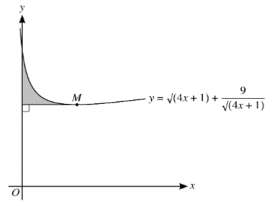


- Area between two curves with respect to y
  - Make x the subject in both equations then integrate its difference
  - Use dy

$$\int_a^b x_1 - x_2\ \mathrm{dy}\ \text{quador} \quad \int_a^b x_1\ \mathrm{dy} - \int_a^b x_2\ \mathrm{dy}$$



{S19-P01} Question 11:



The diagram shows part of the curve:
$y = \sqrt{4x+1} + \frac{9}{\sqrt{4x+1}}$ and the minimum point M.

1. Find expressions for $\frac{\mathrm{dy}}{\mathrm{dx}}$ and $\int y\ \mathrm{dx}$
2. Find the coordinates of M
3. The shaded region is bounded by the curve, the y-axis and the line through M parallel to the x-axis. Find, showing all necessary working, the area of the shaded region.

Solution:

i. Differentiate the equation:

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{d}{\mathrm{dx}}\left(\sqrt{4x+1} + \frac{9}{\sqrt{4x+1}}\right)$$

Use the Chain Rule:

$$\frac{d}{\mathrm{dx}}\left((4x+1)^{\frac{1}{2}} + 9(4x+1)^{-\frac{1}{2}}\right)$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{2}{\sqrt{4x+1}} - \frac{18}{(4x+1)^{\frac{3}{2}}}$$

Integrate the equation:

$$\int y\,dx = \int \sqrt{4x+1} + \frac{9}{\sqrt{4x+1}}\ \mathrm{dx}$$

Apply the reverse chain rule:

$$= \int (4x+1)^{\frac{1}{2}} + 9(4x+1)^{-\frac{1}{2}}\ \ \mathrm{dx}$$

Don't forget to include $+\mathbf{c}$

$$\int \mathbf{y\ dx} = \frac{(\mathbf{4x+1})^{\frac{3}{2}}}{\mathbf{6}} + \frac{\mathbf{9}}{\mathbf{2}}\sqrt{\mathbf{4x+1}} + \mathbf{c}$$

ii. Since M is minimum point, find its coordinates by using $\frac{\mathrm{dy}}{\mathrm{dx}} = 0$

$$\frac{2}{\sqrt{4x+1}} - \frac{18}{(4x+1)^{\frac{3}{2}}} = 0$$

Combine the fractions:

$$\frac{8x-16}{(4x+1)^{\frac{3}{2}}} = 0$$

$$\Rightarrow 8x - 16 = 0$$

$$\Rightarrow x = 2$$

Putting the x-value back to the equation of the curve will give us:

$$\sqrt{4(2)+1}+\frac{9}{\sqrt{4(2)+1}}=6$$

$$\therefore M\,(2,6)$$

iii. The line passing through M is parallel to the x-axis which means its equation is simply:

$$y=6$$

We know that: 1. This is an area between two curves 2. It ranges from $x=0$ to $x=2$ which means our integral will be:

$$\int_0^2 \sqrt{4x+1}+\frac{9}{\sqrt{4x+1}}-6\,dx$$

Which simplifies to:

$$\left[\frac{(4x+1)^{\frac{3}{2}}}{6}+\frac{9}{2}\sqrt{4x+1}-6x\right]_0^2$$

Compute its value

$$\left[\frac{(4x+1)^{\frac{3}{2}}}{6}+\frac{9}{2}\sqrt{4x+1}-6x\right]_0^2=\frac{4}{3}$$

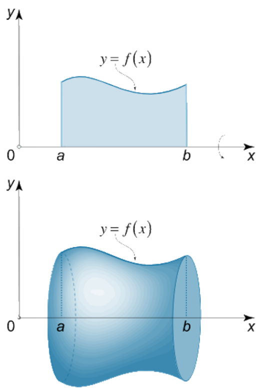
$$\therefore \text{ The area is } \frac{4}{3}$$

Note: You can integrate the two equations separately and then subtract the area, you will still get the same answer

8.4. Volume of Revolution

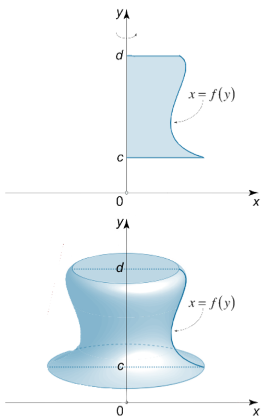
- With respect to x
  - Use $\mathrm{d}x$
  - Make y the subject of the equation of the curve then input πy^2 in the integral

$$\int_a^b \pi y^2\ \mathrm{dx}$$



- With respect to y
  - Use $\mathrm{d}y$
  - Make x the subject of the equation of the curve and input πx^2 in the integral

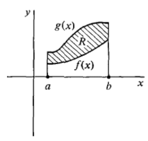
$$\int_a^b \pi x^2\ \mathrm{dy}$$

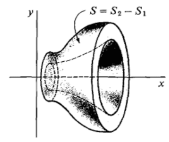


8.5. Volume of Revolution Between 2 Curves

- With respect to x
  - Just like a normal Volume of Revolution, this time we subtract two volumes off each other
  - Use $\mathrm{d}x$
  - Make sure that y is the subject of the equations of the two curves

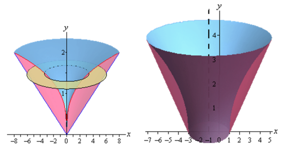
$$\pi\int_a^b y_1^2-y_2^2\,\mathrm{dx}\quad\text{or}\quad\int_a^b \pi y_1^2\,\mathrm{dx}-\int_a^b \pi y_2^2\,\mathrm{dx}$$



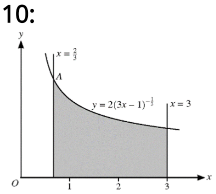


- With respect to y
  - Use $\mathrm{d}y$
  - Make x the subject of the equations of the two curves

$$\pi\int_a^b x_1^2-x_2^2\,\mathrm{dy}\quad\text{or}\quad\int_a^b \pi x_1^2\,\mathrm{dy}-\int_a^b \pi x_2^2\,\mathrm{dy}$$



{W18-P01} Question 10:



The diagram shows part of the curve $y = 2\left(3x - 1\right)^{\frac{1}{3}}$ and the lines $x = \frac{2}{3}$ and $x = 3$. The curve and the line $x = \frac{2}{3}$ intersect at the Point A.
Find, showing all necessary working, the volume obtained when the shaded region is rotated 360° about the x-axis

Solution:
Using the formula for Volume of Revolution:

$$\int_a^b \pi y^2 \ \mathrm{dx}$$

We will get:

$$\int_{\frac{2}{3}}^{3} \pi \left(2\left(3x - 1\right)^{-\frac{1}{3}}\right)^2 \ \mathrm{dx}$$

$$= \int_{\frac{2}{3}}^{3} \pi \left(4\left(3x - 1\right)^{-\frac{2}{3}}\right) \ \mathrm{dx}$$

Integrate it:

$$\left[4\pi\left(3x - 1\right)^{\frac{1}{3}}\right]_{\frac{2}{3}}^{3} = 4$$

CAIE AS LEVEL
Maths (9709)